Optimal Start-Stop Algorithms for Heating & Cooling of Buildings

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1. INTRODUCTION

1.1. Abstract

In this paper we propose a method to model the heating or cooling of a *single-zone insulated space* e.g., room, building. A general approach for modeling the dynamics of a *building's transient thermal response* to active heating or cooling is given.

We provide a summary of commonly used *non-parametric* optimal start-time algorithms. Lastly, we describe a novel *parametric-based* algorithm for estimating the optimal start-time utilizing a *reference-curve* which eliminates the need for analysis based on regression techniques such as described in [1].

1.2. Introduction

To ensure a building's *comfort-level* temperature, *set-point* T_{sp}, is established at the start of the facilities *occupied hours* t_{oc}, building managers often start the HVAC heating/cooling at a *start-time* t_s a fixed time prior to the buildings start of operation.

The *delay-time* t_d is the *pre-conditioning* time needed to overcome the building's *thermal inertia* and ensure acceptable comfort level set-point T_{sp} is reached prior to the start of the occupied period t_{oc}.

As stated, in many cases the start of the pre-conditioning start time t_s is based on a fixed number of hours prior to start of the occupied period t_{oc}. This fixed *start-time* is referred to as *set-back*. This is often the methodology employed when HVAC *Roof Top Units* (RTUs) are installed, where the start sequence is set at installation [2].

Without a priori knowledge of the building's specific thermal characteristics such as: *time-constant, thermal resistance, thermal capacitance*. The facility managers often resort to setting the start-time t_s to a very conservative value to ensure comfort levels are reached at worst case outside air temperature conditions.

Thus, start-times t_s are typically 1 to 2 hours prior to the start of occupancy t_{oc}, depending on the energy capacity of the equipment, thermal characteristics of the building and the geographic temperature profile.

To illustrate; consider a heating system in the Northeastern United States that is designed for worst case outdoor temperature of 0°F, it is obvious the preponderance of days would require a significantly shorter pre-conditioning period t_d due to modest outdoor temperatures relative to worst case conditions. In light of this, there exist an opportunity for energy savings by optimizing the start-time t_s.

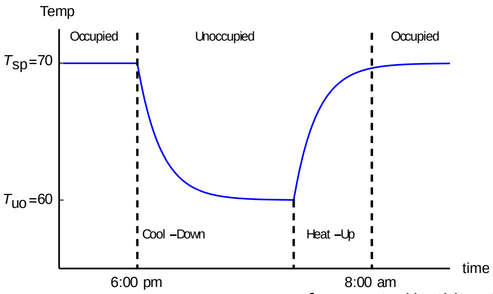
The *optimal start-time algorithms* described in this paper seek to minimize the duration of the pre-conditioning period t_d based on existing indoor and outdoor temperature conditions. That is, it seeks the *minimal pre-conditioning* period t_d necessary to overcome the buildings thermal inertia and thermal losses as a function of outside temperature T_o.

It should be noted that during pre-conditioning, fresh air is not required. Therefore, energy is saved not only from reduced run-time of the heating and cooling equipment but also by reducing the fresh air to a minimum resulting in reduced supply fan power [2].

Although this paper focuses on the heating of a building, the results are easily extended to the cooling of a building. The difference is in the "sign" of the heat generator. For heating, the heat generator is positive $Q_g > 0$, pumping thermal energy into the building. For cooling, the heat generator is negative $Q_g < 0$, pumping thermal energy out of the building.

The following Figure 1-1 illustrates a typical building's heating profile for the occupied and unoccupied periods.

Figure 1-1



Room temp for a typical building (not to scale)

The occupied period is characterized by a constant *comfort-level temperature* referred to as the temperature set-point denoted by T_{sp} .

Comfort level is maintained by the heat source Q_g until the beginning of the unoccupied period at which point the heat source Q_g is turned off. At that point the indoor temperature $T_i(t)$ decays exponentially until it reaches equilibrium with the outdoor temperature T_o or until it reaches the unoccupied set-point temperature T_{uo}

At some point t_s prior to the start of the occupied period the heat source Q_g is turned on and the indoor temperature $T_i(t)$ increases exponentially until it reaches the occupied set-point temperature T_{sp} .

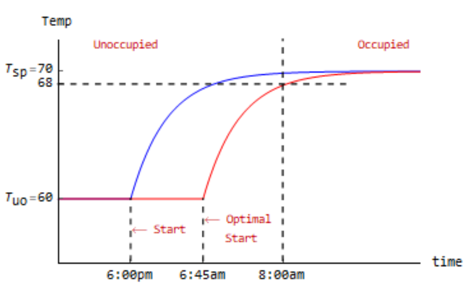


Figure 1-2 Optimal Start

Figure 1-1 shows a hypothetical case illustrating the difference between *optimal* and *fixed start-times*. As shown, with respect to the optimal start-time, the fixed start-time starts 45 minutes too soon.

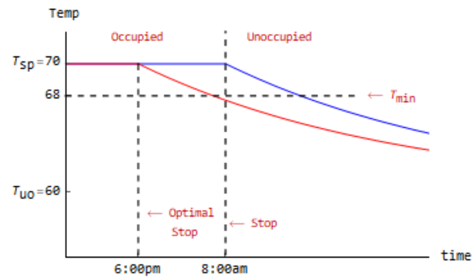


Figure 1-3 Optimal Stop

Figure 1-3 shows a hypothetical case illustrating the difference between *optimal* and *fixed stop-time*. With respect to the optimal stop-time, the fixed stop-time starts 2 hours later than it needs to. The takeaway is that this building's stored heat is not taken advantage of.

1.3. symbols

The following table list symbols and definitions for variables and parameters used in this report[1].

Sym	Unit	Meaning
W	*J/s*	Watt = joules/sec
W_h	*J/Hr*	Watts per hour. We will find it more convineient to work in units of hours rather than seconds
T_{sp}	°*F*	Steady state indoor temperature (set-point) in occupied mode
T_o	°*F*	Outdoor temperature [°F]
$T_i(t)$	°*F*	Internal building temperature at node i
$T_i(0)$	°*F*	Indoor temperature at the start of the *heat-up* mode, $T_{uo} \equiv T_i(0)$
T_{uo}	°*F*	Unoccupied set-point, $T_{uo} \equiv T_i(0)$
Q_g	*W*	Capacity (watts) of the heat generator e.g., boiler, furnace, Air Conditioner
$\bar{Q}_g$	*W*	Average heat flux (watts) into the building
δ	%	Cycle time of the heat source in occupied mode
Q_i	*W*	Heat flux (watts) absorbed by the building's thermal mass e.g., walls, furniture, air
Q_o	*W*	Heat flux (watts) leaving the building i.e., leakage
R_i	°*F/Wh*	Thermal resistance of the building
C_i	*J/°F*	Thermal capacitance of the building.
T_d	*Hrs*	Amount of time required to heat the building from T_{uo} to T_{sp}
τ	*Hrs*	Thermal time-constant of the building $\tau = R_i C_i$ where units are: $units \rightarrow \left[\frac{°F}{Wh}\right]\left[\frac{J}{°F}\right] = \left[\frac{J}{J/Hr}\right] = [Hr]$

Table 1-1 Symbols and Definitions

1 It more convenient to work in units of hours. Define: W-h = joules/hour and thermal resistance R = °F/W-h *and time constant* τ = *hours.*

1.4. Thermal Networks

Recall that most physically realizable systems can have one or more of the following types of elements:

- Energy dissipating
- Potential energy storing
- Kinetic energy storing

It is well known that any system containing only one type of energy *storing* elements can be modeled using first-order differential equations. Any system with both energy *storing* elements is modeled by second-order differential equations.

Typical systems we encounter in real life possess the energy elements described above. Examples of such systems include Electrical, Mechanical, Hydraulic, Pneumatic, Fluid, Acoustic and Magnetic.

It is interesting to note that there is *no element in any thermal system capable of storing kinetic energy*, making thermal systems unique among physical systems. Thermal systems are limited to one or more of the following types of elements: (1) energy dissipating and (2) potential energy storing. The absence of any kinetic energy storing elements means thermal systems are *first-order systems only*.

We facilitate the modeling of a thermal system by use of well-known *thermoelectric analogies* or *thermal networks* [3]. We list the following thermoelectric analogies.

Thermal	Electrical
Building Mass	Capacitor
Temperature	Voltage
Heat Flow	Current
Thermal Resistance	Resistance

Table 1-2 Electromechanical analogies

The thermal network for a *single-zone space* or building that is heated by a heat source such as a boiler or electric heater is shown below.

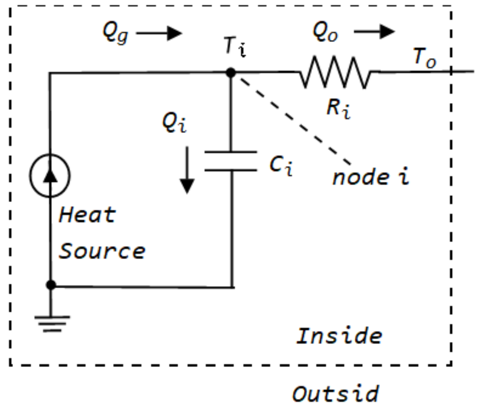


Figure 1-4 Thermal Network

Laplace Transform

In systems analysis one of the most powerful tools available to the system engineer is the *Laplace Transform*. The Laplace transform of a function $f(t)$ is denoted by $\mathcal{L}(f(t))$ or $F(s)$.

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{st}dt \tag{1.1}$$

The Laplace transform is a *linear transform* frequently used to transform a *time-domain* system into to a simpler *complex-frequency domain* system.

The time-domain system is typically comprised of integrals and derivatives as functions of the real variable t, whereas the complex-frequency system is a system of equations comprised of functions of the complex variable $s = \sigma + i\omega$, where $i = \sqrt{-1}$. The complex-frequency system is a system of *algebraic equations*.

The algebraic system is much simpler to manipulate than working directly with the time-domain system. For this reason, system analysis for complex electrical and mechanical systems is almost always done in the complex-frequency domain.

For those familiar with system analysis, to obtain the output response $y(t)$ of an arbitrary system in the time-domain requires the convolution of the input function $f(t)$ with the system impulse response $h(t)$

$$f(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \Leftrightarrow y(t) = f(t) * h(t)$$

Figure 1-5 Time-domain Analysis

In the previous figure the symbol $*$ indicates convolution. Convolution is usually very tedious and complex to execute. On the other hand, to obtain the output $Y(s)$ in the complex-frequency-domain requires multiplying the input function $F(s)$ by the system *transfer function* $H(s)$

$$F(s) \rightarrow \boxed{H(s)} \rightarrow Y(s) = F(s) \times H(s)$$

Figure 1-6 Frequency Domain Analysis

Multiplication is far easier than convolution and this is the principal advantage of analysis in the complex-frequency domain.

To Summarize:

It is desired to obtain the response $y(t)$ of a system with an impulse response $h(t)$ driven by an input function $f(t)$. Figure 1-7 illustrates this process.

$$\begin{array}{ccc} f(t) & & y(t) \\ \downarrow & & \uparrow \\ F(s) \rightarrow & \boxed{H(s)} & \rightarrow F(s) \times H(s) = Y(s) \end{array}$$

Figure 1-7 Analysis using Laplace Transform

First, take the Laplace Transform of the input $f(t) \rightarrow F(s)$ and impulse response $h(t) \rightarrow H(s)$. Next, obtain the system output response $Y(s)$ by multiplying $F(s)$ with $H(s)$. Lastly, compute the Inverse Laplace Transform of $Y(s)$ to obtain the time-domain output response $y(t)$.

The Inverse Laplace transform $Y(s) \rightarrow y(t)$ can be computed by contour integration or by the method of residues, both of which are described in [4].

We will utilize the complex-frequency domain method to analyze the thermal network in Figure 1-4. In this particular case, we will find it necessary to compute the Laplace transform of a derivative and a constant.

The *Laplace transform of a constant* is

$$\mathcal{L}\{k\} = \frac{k}{s} \tag{1.2}$$

Proof

$$\mathcal{L}\{k\} = \int_0^\infty k\, e^{-st} dt = k \int_0^\infty e^{-st} dt \tag{1.3}$$

Substitute $u = -st, \quad du = -s\,dt$ yields

$$\begin{aligned} \mathcal{L}\{k\} &= -\frac{k}{s}\int_0^\infty e^u du = -\frac{k}{s}\left[e^u\right]_0^\infty \\ &= -\frac{k}{s}\left(e^{-\infty} - e^0\right) = \frac{k}{s} \quad Q.E.D. \end{aligned} \tag{1.4}$$

The *Laplace transform of a derivative* is

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{1.5}$$

Proof

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t) e^{-st} dt \tag{1.6}$$

Integration by parts given

$$\begin{aligned} &u = e^{-st}, \quad du = -se^{-st} dt \\ &dv = f'(t)dt, \quad v = f(t) \end{aligned} \tag{1.7}$$

Yields

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\left(-se^{-st} dt\right) \\ &= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty f(t) e^{-st} dt \\ &= e^{-\infty} f(\infty) - e^0 f(0) + s\mathcal{L}\{f(t)\} \end{aligned}$$

From which

$$\mathcal{L}\{f'(t)\} = -f(0) + sF(s) \quad Q.E.D. \tag{1.8}$$

1.5. *Heat-up*

From Figure 1-4 and utilizing the first law of thermodynamics (energy conservation), we can write the balance equations in *heat-up mode* for the space at node i. That is, the heat into node i must equal the heat leaving the node, thus

$$Q_g = Q_i + Q_o \tag{1.9}$$

The heat leakage Q_o is due to heat lost arising from a temperature gradient ΔT across the building's *thermal resistance* R_i.

$$Q_o = \frac{T_i(t) - T_o}{R_i} \tag{1.10}$$

Observe if Q_o is positive then $T_i(t) > T_o(t)$ and heat flows out of the building.

The heat stored by the building's mass is proportional to the *time-rate of change* of indoor temperature $T_i(t)$. The constant of proportionality is known as the *thermal capacitance* C_i and governs how quickly thermal energy can be stored and dissipated by the building mass. Therefore, the heat flux into the building mass is given by

$$Q_i = C_i \frac{d}{dt} T_i(t) \tag{1.11}$$

The pre-conditioning period is relatively short, in the order 2 hours or less. Therefore, we can assume that the outside temperature during the pre-conditioning period is *approximately constant*

$$T_o(t) \approx T_o \tag{1.12}$$

We also assume the heat source is turned on to full capacity with a duty cycle of 100% during *heat-up*. Therefore, based on the energy balance at node i we have the *Ordinary Differential Equation* (ode)

$$C_i \frac{d}{dt} T_i(t) + \frac{T_i(t) - T_o}{R_i} - Q_g = 0 \tag{1.13}$$

Taking the Laplace Transform of the system ode yields

$$T_i(s) = -C_i \left(sT_i(s) - T_i(0) \right) + \frac{Q_g}{s} - \frac{T_o}{sR_i} - \frac{T_i(s)}{R_i} \tag{1.14}$$

We factor the denominator of $T_i(s)$ as a product of its *poles* in the following form.

$$T_i(s) = \frac{f(s)}{(s+k_i)(s+k_i)\dots(s+k_i)} \tag{1.15}$$

Factoring (1.14) yields

$$\begin{aligned} T_i(s) &= \frac{s\, R_i C_i T_i(0) + Q_g R_i + T_o}{s(sR_iC_i + 1)} \\ &= \frac{s\, T_i(0) + Q_g / C_i + T_o / \tau}{s(s + 1/\tau)} \end{aligned} \tag{1.16}$$

Where $\tau = R_i C_i$.

The *transient-response* is found by taking the inverse Laplace Transform of $T_i(s)$. The inverse Laplace Transform of (1.16) in this factored form can be computed by summing the residues as described in [4] where the definition of the residue for a *simple first-order pole* at p_1 is

$$res(p_1) = \lim_{s \to -p_1} \frac{f(s)(s+p_1)e^{st}}{(s+p_1)(s+p_2)\dots(s+p_i)} \tag{1.17}$$

The poles in (1.16) are $p_1 = 0$, $p_2 = 1/\tau$ from which the residues for $T_i(s)$ are

- The residue at $p_1 = 0$ is

$$\begin{aligned} \operatorname{res}(0) &= \lim_{s \to 0} T_i(s)(s+0)e^{st} \\ &= \lim_{s \to 0} \frac{s\, T_i(0) + Q_g / C_i + T_o/\tau}{s(s + 1/\tau)} s e^{st} \\ &= \frac{Q_g / C_i + T_o/\tau}{1/\tau} = R_i Q_g + T_o \end{aligned} \tag{1.18}$$

- The residue at $p_2 = 1/\tau$ is

$$\begin{aligned} \operatorname{res}(1/\tau) &= \lim_{s \to -1/\tau} T_i(s)(s + 1/\tau)e^{st} \\ &= \lim_{s \to -1/\tau} \frac{s\, T_i(0) + Q_g / C_i + T_o/\tau}{s(s+1/\tau)} (s + 1/\tau) e^{st} \\ &= \frac{-T_i(0)/\tau + Q_g / C_i + T_o/\tau}{-1/\tau} e^{-t/\tau} \\ &= \left(T_i(0) - R_i Q_g - T_o \right) e^{-t/\tau} \end{aligned} \tag{1.19}$$

Upon summing the residues, the transient *thermal response* for the *heat-up* mode is

$$T_{hu}(t) = e^{-t/\tau} \left(T_i(0) - R_i Q_g - T_o \right) + R_i Q_g + T_o \ [hrs] \tag{1.20}$$

1.6. *Cool-down* period

In the *cool-down* period the heat generator is off, $Q_g = 0$ and the discharge heat flux given up by the thermal mass equals the negative of the leakage heat flux. That is, $Q_i = -Q_o$ from which the system ode is

$$C_i \frac{d}{dt} T_i(t) + \frac{T_i(t) - T_o}{R_i} = 0 \tag{1.21}$$

Taking the Laplace Transform of the system ode yields

$$T_i(s) = C_i \left(sT_i(s) - T_i(0) \right) - \frac{T_o}{sR_i} + \frac{T_i(s)}{R_i} \tag{1.22}$$

Initially at the beginning of *cool-down*, the indoor temperature is at set point, so set $T_i(0) = T_{sp}$ then solve for $T_i(s)$ and factor the denominator of $T_i(s)$ as a product of its *poles* from which

$$T_i(s) = \frac{sC_iR_iT_{sp} + T_o}{s(sR_iC_i + 1)} = \frac{sT_{sp} + T_o/\tau}{s(s + 1/\tau)} \tag{1.23}$$

The poles in (1.23) are $p_1 = 0$, $p_2 = 1/\tau$ from which the residues for $T_i(s)$ are:

- The residue at $p_1 = 0$ is

$$\begin{aligned} \operatorname{res}(0) &= \lim_{s\to 0} T_i(s)(s+0)e^{st} \\ &= \lim_{s\to 0} \frac{(sT_{sp} + T_o/\tau)se^{st}}{s(s+1/\tau)} \\ &= \frac{(0T_{sp} + T_o/\tau)e^{0t}}{(0+1/\tau)} = T_o \end{aligned} \tag{1.24}$$

- The residue at $p_2 = 1/\tau$ is

$$\begin{aligned} \operatorname{res}(1/\tau) &= \lim_{s\to -1/\tau} T_i(s)(s+1/\tau)e^{st} \\ &= \lim_{s\to -1/\tau} \frac{(sT_{sp} + T_o/\tau)(s+1/\tau)e^{st}}{s(s+1/\tau)} \\ &= \frac{(-T_{sp}/\tau + T_o/\tau)e^{-t/\tau}}{-1/\tau} \\ &= e^{-t/\tau}(T_{sp} + T_o) \end{aligned} \tag{1.25}$$

Upon summing the residues, the transient *thermal response* for the *cool-down* mode is

$$T_i(t) = e^{-t/\tau}(T_{sp} + T_o) + T_o \quad [hrs] \tag{1.26}$$

1.7. Thermal Time-constant

The buildings thermal time-constant τ can be determined from the *cool-down* period. Given the *cool-down* equation (1.26)

$$T_i(t) = e^{-t/\tau}(T_{sp} + T_o) + T_o \tag{1.27}$$

Rearranging

$$e^{-t/\tau} = \frac{T_i(t) - T_o}{T_{sp} - T_o} \tag{1.28}$$

Taking the log of both sides

$$-\frac{t}{\tau} = \ln\left(\frac{T_i(t) - T_o}{T_{sp} - T_o}\right) \tag{1.29}$$

Solving for τ

$$\tau = \frac{t}{\ln\left(\frac{T_i(t) - T_o}{T_{sp} - T_o}\right)} \tag{1.30}$$

Since t is arbitrary, we can define a test point t_x from which the *thermal time-constant* is

$$\tau(t_x) = \frac{t_x}{\ln\left(\frac{T_i(t_x) - T_o}{T_{sp} - T_o}\right)} \tag{1.31}$$

Example 1-1

Given a system with $\tau = 5$, $T_{sp} = 70°$ and $T_O = 5°$. Using *cool-down*, find the thermal time-constant τ.

Solution

We start by plotting the actual system for validation.

$$sub = \{\tau = 5,\ T_{sp} = 70,\ T_o = 5\}$$

$$\begin{aligned} T_{cd}(t) &= e^{-t/\tau}(T_{sp} + T_o) + T_o \leftarrow sub \\ &= e^{-t/5}(70-5) + 5 \\ &= 65e^{-t/5} + 5 \end{aligned} \tag{1.32}$$

The plot of the actual system $T_i(t)$ is shown below.

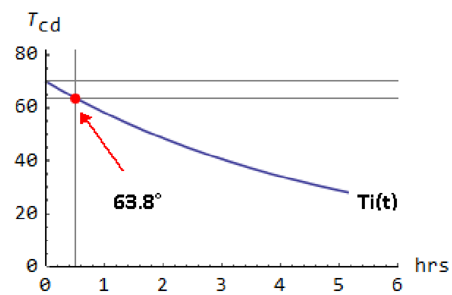


Figure 1-8 System *cool-down*

Substituting $t = 0.5$ into (1.32) gives the observed *cool-down* temperature at our test point.

$$T_i(0.5) = 65e^{-0.5/5} + 5 = 63.8\ \mathrm{F}^{\circ} \tag{1.33}$$

Based only on our observed test-point and the following parameters we solve for τ.

$$sub = \{t_x = 0.5,\ T_{sp} = 70^{\circ},\ T_o = 5^{\circ}\}$$

$$\tau = \frac{t_x}{\ln\left(\frac{T_i(t_x) - T_o}{T_{sp} - T_o}\right)} \leftarrow sub = \frac{0.5}{\ln\left(\frac{63.8-5}{70-5}\right)} = 4.99\ hrs$$

From which the *time-constant* is 4.99 hours as expected

1.8. Thermal Resistance

During the *occupied mode*, the heat source Q_g will toggle on and off providing the *average* heat flux $\bar{Q}_g$ necessary to maintain temperature set-point T_{sp}

$$\bar{Q}_g = Q_g \frac{t_{on}}{t_{off}} = \delta Q_g \tag{1.34}$$

If we measure the average heat flux $\bar{Q}_q = \delta Q_q$ at steady state, the average internal temperature is $T_i(t) = T_{sp}$ then the *thermal resistance* is given by

$$R_i = \frac{T_i(t) - T_o}{\delta Q_g} = \frac{T_{sp} - T_o}{\delta Q_g} \left[\frac{F^\circ}{Wh}\right] \tag{1.35}$$

Example 1-2

A space is heated with a source $\delta Q_g = 500\ Wh$. At steady state the heat source cycle time is $\delta = 50\%$. The indoor set-point is $T_{sp} = 70^\circ$ and the outdoor temperature is $T_o = 5^\circ$. Find the thermal resistance for the space R_i .

Solution

$$subs = \{T_{sp} = 70,\ T_o = 5,\ Q_g = 500,\ \delta = 0.5\}$$

$$R_i = \frac{T_{sp} - T_o}{\delta Q_g} \leftarrow subs = 0.26\ [{}^\circ F / Wh] \tag{1.36}$$

1.9. Thermal Capacitance

Given we have calculated the thermal *time-constant* τ and thermal resistance R_i as described above we can now calculate the thermal capacitance C_i as follows

$$C_i = \frac{\tau}{R_i}\left[Hr \frac{J/Hr}{F^\circ}\right] \to \left[\frac{J}{{}^\circ F}\right] \tag{1.37}$$

We can define the thermal *time-constant* in terms of thermal resistance and capacitance.

$$\tau = R_i\, C_i \tag{1.38}$$

As a check, we see the units of the *time-constant* is hrs.

$$\tau = R_i\, C_i \to \left[\frac{F^\circ}{Wh}\right]\left[\frac{Wh - Hr}{F^\circ}\right] \to [Hr]$$

Example 1-3

Given $R_i = 0.26$ from the previous example find C_i

Solution

$$\begin{aligned} subs &= \{\tau = 5, R_i = 0.26\} \\ C_i &= \frac{\tau}{R_i} \leftarrow subs = 19.2308 \end{aligned} \tag{1.39}$$

1.10. Heat-Up (As a Function of T_{od})

The space will be heated until the system has reached steady-state. At this point the heat source balances the heat loss $\delta Q_g = (T_{sp} - T_{od}) / R_i$. From which the heat source is

$$Q_g = \frac{T_{sp} - T_{od}}{\delta R_i} \tag{1.40}$$

By equation (1.20)

$$\begin{aligned} T_{hu}(t) = &-e^{-t/\tau}\left(Q_g R_i - T_i(0) + T_o\right) \\ &+ Q_g R_i + T_o \end{aligned} \tag{1.41}$$

Substituting which $\delta Q_g = (T_{sp} - T_{od}) / R_i$ into $T_{hu}(t)$ yields the transient thermal response as a function of the design temperature T_{od} .

$$T_{hu}(t) = -e^{-t/\tau}\left(-T_i(0) + T_o + \frac{T_{sp} - T_{od}}{\delta}\right) + T_o + \frac{T_{sp} - T_{od}}{\delta} \tag{1.42}$$

Example 1-4

Assume we have designed a heat source for worst case outdoor temperature $T_{od} = 0^\circ$ and we have measured a cycle-time $\delta = 1$. Plot the transient response to heat the space to steady state if the outdoor temperature is also $T_o = 0^\circ$.

Solution

$$subs : \begin{Bmatrix} \tau = 5,\ T_{sp} = 70,\ Ti(0) = 60 \\ T_o = 0,\ T_{od} = 0,\ \delta = 1 \end{Bmatrix}$$

$$T_{hu}(t) \leftarrow subs = 70 - 10\, e^{-t/5}$$

We observe that the steady state temperature is 70° as expected.

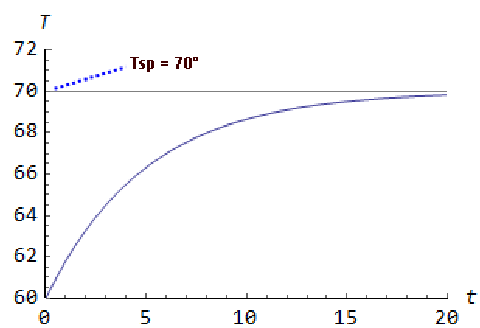


Figure 1-9 Slow rise-time

Example 1-5

Assume we have a heat source designed for worst case outdoor temperature $T_{od} = 0^\circ$ and we have measured a cycle-time $\delta = 1$. Plot the transient response to heat the space to steady state if the outdoor temperature is $T_o = 30^\circ$.

Solution

$$\begin{aligned} &subs : \begin{Bmatrix} \tau = 5,\ T_{sp} = 70,\ Ti(0) = 60 \\ T_o = 30,\ T_{od} = 0,\ \delta = 1 \end{Bmatrix} \\ &T_{hu}(t) \leftarrow subs = 100 - 10\, e^{-t/5} \end{aligned} \tag{1.43}$$

We observe the temperature reaches 70° very quickly and rises approximately linearly.

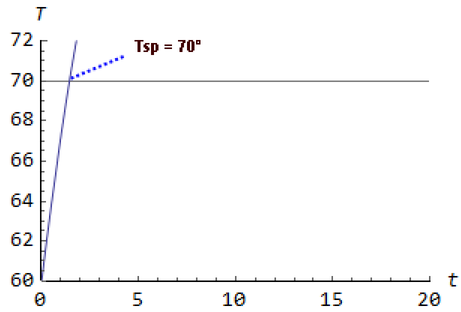


Figure 1-10 Fast rise-time

1.11. Estimating Delay-time

If we know the heat capacity of the average heat source δQ_g required to sustain steady state, we can estimate the time required to heat the space from $T_i(0)$ to T_{sp} given we know the time-constant τ, the thermal resistance R_i and outside temperature T_o.

The *delay-time* t_d is the time the heat must be turned on before the start of the occupied period in order for the space to be heated to the set-point temperature T_{sp} just in time for the start of the occupation period.

Given the equation for *heat-up*

$$T_{hu}(t) = -e^{-t/\tau}\left(Q_g R_i - T_i(0) + T_o\right) + Q_g R_i + T_o \tag{1.44}$$

Setting $T_{hu}(t) = T_{sp}$ and solving for t_d gives the delay-time as

$$t_d = \tau \log\left(\frac{Q_g R_i - T_i(0) + T_o}{Q_g R_i + T_o - T_{sp}}\right) \tag{1.45}$$

If the start of the occupied period is t_{oc} then the *optimal start-time* t_s is given by $t_s = t_{oc} - t_d$ from which the start-time is given by

$$t_s = t_{oc} - \tau \log\left(\frac{Q_g R_i - T_i(0) + T_o}{Q_g R_i + T_o - T_{sp}}\right) \tag{1.46}$$

Example 1-6

Assume we have designed a heat source for worst case outdoor temperature $T_{od} = 0°$ and the current outdoor temperature is $T_o = 10°$. Plot the transient response to heat the space to steady state and calculate the t_d required to heat the space to T_{sp}.

Solution

The system parameters are

$$subs = \begin{Bmatrix} \tau = 5,\ Q_g = 250,\ R_i = 0.3, \\ T_i(0) = 60,\ T_o = 10,\ T_{sp} = 70 \end{Bmatrix}$$

Substituting into the *heat-up* equation (1.20) and the delay equation delay equation (1.45) yields

$$T_{hu}(t) \leftarrow subs = 85 - 25e^{-t/5}$$
$$t_d = 2.554$$

A plot of $T_{hu}(t)$ indicates, the time required to increase the initial internal space temperature from $T_i(0) = 60°$ to set-point $T_{sp} = 70°$ is $t_d = 2.55$ hrs.

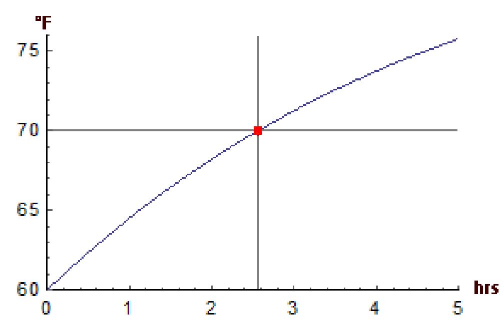


Figure 1-11

2. NON-PARAMETRIC ALGORITHMS

2.1. Fixed Start (Setback)

The *fixed setback* is the simplest form of a *non-optimal* start algorithm. The Building Management System (BMS) will start the HVAC system at a fixed number of hours before occupancy as specified by the user [5]. Typically, this method is set once unless an exception day occurs. An *exception day* is a day where the pre-conditioning time t_d is too short. In this case the system manager would *ratchet back* the *start-time* to ensure the set-point is met under worst case conditions. This is *not an optimal* start algorithm.

2.2. Constant Temp Gradient

A very simple adaptive start algorithm is based on a *Constant Temperature Gradient* that remains essentially the same from day to day [5]. In this case, the temperature gradient $s = \Delta T / hr$ is measured. The difference between the unoccupied space temperature T_{uo} and the occupied set-point temperature T_{sp} is divided by s to calculate the actual number of hours t_d for pre-conditioning.

ALGORITHM:

Initializing:

1. Measure rise-time $s = \Delta T / hr$
2. Calculate the delay time
   $t_d = \left(T_{sp} - T_{uo}\right) / s$
3. Set start time $t_s = t_{oc} - t_d$

Periodic:

4. Periodically initialize

2.3. Adaptive Temp Gradient

The *Adaptive Temperature Gradient* algorithm takes the arithmetic average of the actual temperature gradients calculated for a specified number of previous n days.

Given s_n is the rise-time on the *nth* day, a recursive algorithm can be used to calculate the average of the past n rise-times in an iterative fashion.

Defining the vector $S = \left\{s_0, s_1 \cdots s_n\right\}$ as the set of the past n rise-times. The mean of the vector S denoted $\left\langle s \right\rangle_n$ can be found recursively as follows:

$$\langle S \rangle_n = \frac{n-1}{n}\langle S \rangle_{n-1} + \frac{1}{n} s_n \tag{1.47}$$

Proof

$$\begin{aligned}
\langle S \rangle_n &= \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n-1} s_i + \frac{1}{n} s_n \\
&= \frac{n-1}{n}\,\frac{1}{n-1}\sum_{i=1}^{n-1} s_i + \frac{1}{n} s_n \\
&= \frac{n-1}{n}\langle S \rangle_{n-1} + \frac{1}{n} s_n \quad \text{Q.E.D.}
\end{aligned}$$

ALGORITHM:

Initializing:

1. Measure rise-time $s = \Delta T / Hr$
2. Calculate the delay time
   $t_d = \left(T_{sp} - T_{uo}\right) \div s$
3. Set start time $t_s = t_{oc} - t_d$

Daily Events:

1. Measure rise-time s_n for the current (nth) day
2. Measure the average rise-time $\left\langle s \right\rangle_n$ for the vector $S = \left\{s_0, s_1 \ldots s_n\right\}$ where:

   $$\langle S \rangle_n = \frac{n-1}{n}\langle S \rangle_{n-1} + \frac{1}{n} S_n$$

3. Calculate $t_d = \left(T_{sp} - T_{uo}\right) \div \left\langle s \right\rangle_n$
4. Set $t_s = t_{oc} - t_d$

2.4. Table Lookup

This method requires measuring daily *delay-times* t_d and outside air temperatures T_o. The recordings are kept in a data base with one or two-degree bins e.g., $30^\circ \geq T < 32^\circ$, $32^\circ \geq T < 34^\circ$ etc. Eventually a complete history of delay-times vs. outside air temp will be accumulated. Then it is a simple matter of table-lookup.

ALGORITHM

Initialization

1. Set delay-time t_d to worst case
2. Create a data base with two-degree temp-bins

Daily Events

1. Measure T_o and look up $t_d(bin)$.
2. Set $t_s = t_{oc} - t_d(bin)$
3. Measure actual t_d and update the temp-bin using $Max\left(t_d, t_d(bin)\right)$

3. PARAMETRIC ALGORITHMS

3.1. Parametric Method

Parametric algorithms use the *thermal parameters* of a building such as: thermal *time-constant* τ ,thermal *resistance* R_i and thermal *capacitance* C_i .

We can calculate a building's thermal time-constant τ using the method described in section 1.7 and thermal resistance R_i using the method described in section 1.8. If we know Q_g , then we can measure T_o and $T_i(0)$ to determine the optimal start-time from (1.46)

$$t_s = t_{oc} - \tau \log\left(\frac{Q_g R_i - T_i(0) + T_o}{Q_g R_i + T_o - T_{sp}}\right) \tag{1.48}$$

3.2. Reference Curve Method (RCM)

As noted in 3.1, the parametric method requires explicit knowledge of Q_g for the space in question. This may not be known, since Q_g may be supplying multiple zones. If we do not know Q_g we can still determine the start-time by parametric means utilizing a novel *Reference Curve Method* (RCM) developed by the author.

In this section we describe the RCM algorithm for predicting the indoor temperature curve. This provides an alternative to regression or temperature gradient techniques. The principal advantage is that it eliminates the need to update a running data-base and perform regression calculations as described in [1].

The idea stems from the observation that in *heat-up* mode, the rise times between any two arbitrary days does not depend on Q_g , T_{sp} , $T_i(0)$, since all these parameters are common (same value) to both days. Only T_o differs. So, start by defining the temperature curve for two arbitrary days

$$\begin{aligned} T_1 = -\mathrm{e}^{-t/\tau}&\left(Q_g R_i - T_i(0) + T_{o1}\right) \\ &+ Q_g R_i + T_{o1} \end{aligned} \tag{1.49}$$

$$\begin{aligned} T_2 = -\mathrm{e}^{-t/\tau}&\left(Q_g R_i - T_i(0) + T_{o2}\right) \\ &+ Q_g R_i + T_{o2} \end{aligned} \tag{1.50}$$

Now we define the difference between the two curves

$$T_{12} = T_1 - T_2 \rightarrow T_2 = T_1 - T_{12} \tag{1.51}$$

From which

$$T_{12} = \mathrm{e}^{-t/\tau}\left(\mathrm{e}^{t/\tau} - 1\right)\left(T_{o1} - T_{o2}\right) \tag{1.52}$$

From which T_2 is determined by subtracting T_{12} from T_1

- $T_2 = T_1 - T_{12}$ (1.53)

Observe that the difference between any two temperature curves is a function of only τ and the outdoor temperatures T_{o1} for the target day and T_{o2} for an arbitrary test date. No explicit knowledge of Q_g is needed.

First, we first measure the indoor temperature curve by taking a temperature reading every 10 or 15 minutes and store the curve values in a database.

Second, we calculate τ and measure T_{o1} and store results in memory. Now based on this stored reference curve, τ and T_{o1} we can then determine the temperature curve for *any day* based solely of the knowledge of these stored values and outside temperature for the target day T_{o2} .

This algorithm eliminates the need for table look-up or regression estimates which take many months to establish and does not require regression calculations.

ALGORITHM (REFERENCE CURVE)

Initialization (create reference data)

1. Set delay time t_d to worst case
2. Measure T_{o1} and store in memory
3. Calculate τ and store in memory
4. Measure indoor temperature $T_1(t)$ in 5-minute increments, store in memory i.e., Table 3-1

Daily Events

5. Measure $T_{o2}(t)$
6. Calculate $T_{12}(t)$ and populate table
7. Find t in the row of the table where $T_{12}(t) - T_1 = T_2$
8. Set $t_d = t$

Example 3-1

Assume we have a building with the following parameters: the outside temperature on the *test day* is $T_{o1} = 5°$ and the outside temperature on the *target day* is $T_{o2} = 30°$. We will attempt to generate an estimate of the temperature curve for the target day based solely on the reference curve and T_{o1} .

Solution

Applying the algorithm

1. Measure $T_{o2}(t) = 70°$
2. Calculate $T_{12}(t)$ and populate table
3. Find t in the row of the table where
   $T_{12}(t) - T_1 = T_2 = 70° \rightarrow t = 1.5\,hr$
4. Set $t_d = 1.5\,hr$

From the following table we see that for the current example, the target day T_2 reaches 70° in 1.5 hours.

t_d	Store T_1	Calculate T_{12}	T_2(est)
0.0	60.0	0.0	60.0
0.1	60.4	-0.2	60.7
0.2	60.9	-0.5	61.4
0.3	61.3	-0.7	62.1
0.4	61.8	-1.0	62.7
0.5	62.2	-1.2	63.4
0.6	62.6	-1.5	64.1
0.7	63.0	-1.7	64.7
0.8	63.5	-1.9	65.4
0.9	63.9	-2.2	66.0
1.0	64.3	-2.4	66.7
1.1	64.7	-2.6	67.3
1.2	65.1	-2.8	67.9
1.3	65.5	-3.0	68.5
1.4	65.9	-3.3	69.1
1.5	66.3	-3.5	69.8
1.6	66.7	-3.7	70.3
1.7	67.0	-3.9	70.9
1.8	67.4	-4.1	71.5
1.9	67.8	-4.3	72.1
2.0	68.2	-4.5	72.7
2.1	68.5	-4.7	73.3
2.2	68.9	-4.9	73.8
2.3	69.2	-5.1	74.4
2.4	69.6	-5.3	74.9
2.5	70.0	-5.5	75.5
2.6	70.3	-5.7	76.0
2.7	70.6	-5.9	76.6
2.8	71.0	-6.1	77.1
2.9	71.3	-6.3	77.6
3.0	71.7	-6.5	78.1

Table 3-1 Reference Curve Memory

To verify the results, we will plot the theoretical curves. Given the reference curves

$$T_1 = -e^{-t/\tau}\left(Q_g R_i - T_i(0) + T_{o1}\right) + Q_g R_i + T_{o1} \tag{1.49}$$

$$T_2 = -e^{-t/\tau}\left(Q_g R_i - T_i(0) + T_{o2}\right) + Q_g R_i + T_{o2} \tag{1.50}$$

Substituting system parameters for the reference curve

$$subs = \left\{ \begin{array}{l} \tau \rightarrow 10,\ Q_g \rightarrow 500,\ R_i \rightarrow 0.2, \\ T_i(0) = 60,\ T_{o1} = 5,\ T_{sp} = 70,\ T_{o2} = 30 \end{array} \right\}$$

For the reference day we have the following temperature curves

$$\begin{aligned} T_1 &\leftarrow subs = 105 - 45\ e^{-t/10} \\ T_2 &\leftarrow subs = 130 - 70\ e^{-t/10} \end{aligned} \tag{1.54}$$

The estimate for the temperature curve on the *target day* is $T_2 = T_1 - T_{12}$ from which

$$T_2(est) = 130 - 70\ e^{-t/10} \tag{1.55}$$

We observe that the *estimated* temperature curve for the target day matches the *actual* temperature for the target day at 70°. Thus, we were able to predict the actual temperature pre-conditioning time t_d for the target day with a knowledge of our reference curve.

Since the reference curve is measured only once at initialization, the only parameter needed to be measured on a daily basis is T_{o2} .

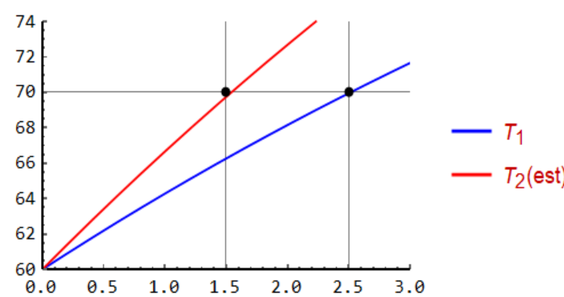


Figure 3-1

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